AD-A116 913 MARYLAND UNIV EASTERN SHORE PRINCESS ANNE F/G 20/4
TWO DIMENSIONAL VISCOUS FLOW OF A COMPRESSIBLE FLUID.(U)
OCT 81 K GOVINDARAJU DAAG29-80-G-0011
UNCLASSIFIED ARO-17610.1-MA-H NL

1 OF 1
AD A
116913

END
DATE
FILMED
08-82
DTIC

ARO 17610.1-MA-H

12

TWO DIMENSIONAL VISCOUS FLOW OF A COMPRESSIBLE FLUID

Kamesh Govindaraju
UNIVERSITY OF MARYLAND EASTERN SHORE
Princess Anne, MD.

RESEARCH supported under contract DAAG29-80-G-0011
of the Army Research Office with the University of Maryland,
Eastern Shore

20 Oct. 1981.

Report No. 1

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1	2. GOVT ACCESSION NO. AD-A116913	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) TWODIMENSIONAL VISCOUS FLOW OF A COMPRESSIBLE FLUID		5. TYPE OF REPORT & PERIOD COVERED FINAL. SEPT.80 - july 81.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) DR. KAMESH GOVINDARAJU		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-g-0011
9. PERFORMING ORGANIZATION NAME AND ADDRESS UNIVERSITY OF MARYLAND-EASTERN SHORE PRINCESS ANNE, MD.21853		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE OCt. 20 1981
		13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

TWODIMENSIONAL FLOWS, VISCOSITY, VORTICITY, PRESSURE, DENSITY NAVIER_STOKES EQUATIONS, STREAM FUNCTION, COMPRESSIBLE FLUID.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Navier-Stokes equations for the twodimensional steady motion of a viscous compressible fluid are transformed to a system of equations in which the streamfunction is one of the two independent coordinates while the second coordinate is arbitrary.

Vorticity, speed, density, pressure and the velocity gradient are the unknown functions of the equations. Three systems of equations

(Contd)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

BLOCK 20:

are developed, all of them underdeveloped. The equations are applied to flows with pre-assigned forms and it is shown that if the stream lines are are straight lines, they must be either parallel or concurrent and if the stream lines are involutes of a curve, then they must be concentric circles.

INDEX

TWODIMENSIONAL VISCOUS FLOW OF A COMPRESSIBLE FLUID

1. PRELIMINARIES

The equations of motion for a twodimensional flow of a compressible viscous fluid are

$$\rho(uu_x+vu_y)+p_x = \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) ,$$

(Navier-Stokes)

(1.1)

$$\rho(uv_x+vv_y)+p_y = \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) ,$$

$$(\rho u)_x + (\rho v)_y = 0 , \qquad \text{(continuity)}$$

Where u and v are the velocity componants, p is the pressure, ρ is the density and μ is the constant coefficient of viscosity. In order to make the system determinate, one assumes that either the density function $\rho(x,y)$ is known, or that the equation of state in the form of a pressure-density relation is known.

We shall develop equivalent systems of equations which replace (1.1) similar to those developed by Martin {1} for viscous incompressible fluids, by Chanda {2} for Cosserat fluids and Nath {3} for magneto-hydrodynamic flows. The new systems of equations will have the vorticity ω, speed q, velocity gradient ε, defined as,

(1.2)

$$\omega = v_x - u_y, \qquad q = \sqrt{u^2+v^2}, \qquad \varepsilon = u_x + v_y,$$

the pressure p and density ρ as dependent variables. A system of curvilinear coordinates defined by the streamfunction ψ and an arbitrary function ϕ form the independent variables of the system. Once solutions are obtained for these equations, the flow may be mapped onto the physical plane.

After establishing the new systems of equations, we shall consider two special cases of flows, one with streamlines enveloping an arbitrary curve and the second case with streamlines as involutes of an arbitrary curve. In both cases we shall show that the curve must reduce to a single point.

In order to arrive at the systems of equations mentioned above, we need the following preliminary results:

We first re-write the system (1.1) as

$$\rho q q_x + p_x - \rho v \omega = -\mu \omega_y + \frac{4}{3}\mu \varepsilon_x \qquad \text{(Navier-Stokes)}$$

$$\rho q q_y + p_y + \rho u \omega = \mu \omega_x + \frac{4}{3}\mu \varepsilon_y \tag{1.3}$$

$$\rho \varepsilon + \rho_x u + \rho_y v = 0 \qquad \text{(continuity)}$$

$$\omega = v_x - u_y, \qquad \varepsilon = u_x + v_y \qquad \text{(Vorticity, Gradient)}$$

The stream function $\psi(x,y)$ is defined by the continuity equation of (1.1),

$$\psi_x = -\rho v, \qquad \psi_y = \rho u \tag{1.4}$$

Let $\phi(x,y)$ be an arbitrary function so that the two systems of curves $\phi = \text{const.}$, $\psi = \text{const.}$ determine a curvilinear coordinate system. We may now treat the variables x, y, u, v, ω, q and ε as functions depending on ϕ, ψ. Once these functions are known in the (ϕ,ψ) - plane, we may transform them to the physical plane provided the Jacobian

$$J = x_\phi y_\psi - x_\psi y_\phi, \tag{1.5}$$

in non-zero and finite. Then

$$x_\phi = J\psi_y, \quad y_\phi = J\psi_x, \quad x_\psi = J\Phi_y, \quad y_\psi = J\Phi_x. \tag{1.6}$$

The first fundamental form for the (x,y)-plane is given by

$$ds^2 = Ed\phi^2 + 2Fd\phi d\psi + Gd\psi^2, \tag{1.7}$$

where

$$E = x_\phi^2+y_\phi^2, \quad F = x_\phi x_\psi+y_\phi y_\psi, \quad G = x_\psi^2+y_\psi^2 \quad \text{and} \quad W = +\sqrt{EG-F^2}. \tag{1.8}$$

The Gaussian curvature given by

$$K = \frac{1^2}{W}\left\{\left(\frac{W}{E}\Gamma_{11}^{2}\right)_\psi - \left(\frac{W}{E}\Gamma_{12}^{2}\right)_\phi\right\}, \tag{1.8}$$

where

$$\Gamma_{11}^{2} = \frac{-FE_\phi+2EF_\phi-EE_\psi}{2W^2},$$

$$\Gamma_{12}^{2} = \frac{EG_\phi-FE_\psi}{2W^2}, \tag{1.9}$$

$$\Gamma_{22}^{2} = \frac{EG_\psi-2FF_\psi+FG_\phi}{2W^2},$$

are the christaffel symbols {4}, should vanish.

If δ is the angle of inclination of the tangent to the streamline ψ = const. in the sense of increasing ϕ, then

$$\delta = \int\frac{J}{E}\,(\Gamma_{11}^{2}d\phi + \Gamma_{12}^{2}d\psi), \tag{1.10}$$

and

$$Z = x+iy = \int\frac{e^{i\delta}}{\sqrt{E}}\{Ed\phi + (F+iJ)d\psi\} \tag{1.11}$$

with $J = \pm W$.

The identities

$$\left(\frac{E}{2W^2}\right)_\phi = \frac{F}{W^2}\Gamma_{11}^{2} - \frac{E}{W^2}\Gamma_{12}^{2}, \tag{1.12}$$

$$(\frac{E}{2W^2})_\psi = \frac{F}{W^2}\ \Gamma_{12}^{\ 2} - \frac{E}{W^2}\ \Gamma_{22}^{\ 2},$$

and

$$(\frac{F}{W})_\phi - (\frac{E}{W})_\psi = \frac{G}{W}\ \Gamma_{11}^{\ 2} - 2\frac{F}{W}\ \Gamma_{12}^{\ 2} + \frac{E}{W}\ \Gamma_{22}^{\ 2} \tag{1.13}$$

will be of use in later calculations.

2. EQUATIONS

With the introduction of vorticity ω, speed q and gradient ε as given in (1.2), the Navier-Stokes equations (1.1) may be written with curvilinear coordinates ϕ, ψ as

$$\rho q q_\phi = \frac{4}{3}\mu\varepsilon_\phi + \mu\omega_\phi \frac{F}{J} - \mu\omega_\psi \frac{E}{J} - P_\phi,$$

$$\rho q q_\psi = \frac{4}{3}\mu\varepsilon_\psi + \mu\omega_\phi \frac{G}{J} - \mu\omega_\psi \frac{F}{J} - P_\psi, \tag{2.1}$$

or, equivalently,

$$G(\rho q q_\phi + \rho_\phi + \frac{4}{3}\mu\varepsilon_\phi) - F(\rho q q_\psi + P_\psi + \omega + \frac{4}{3}\mu\varepsilon_\psi) = \mu J\omega_\phi .$$

$$-F(\rho q q_\phi + \rho_\phi + \frac{4}{3}\mu\varepsilon\phi) + E\ (\rho q q_\phi + P_\psi + \omega + \frac{4}{3}\mu\varepsilon_\psi) = \mu J\omega_\phi . \tag{2.2}$$

We may use theorem 4.1 of {1} to write the equation of continuity as

$$q = \frac{\sqrt{E}}{\rho W} . \tag{2.3}$$

Further we may use (1.6) together with (1.4) to eliminate the velocity componants in the equation of continuity of (1.3) to obtain

$$\varepsilon = \frac{1}{J}\left(\frac{1}{\rho}\right)_\phi , \tag{2.4}$$

or, using (2.3)

$$\frac{E}{J} = -\frac{\rho_\phi q^2}{\varepsilon} . \tag{2.5}$$

Thus the equation of continuity and the equation for the velocity gradient ε may be replaced by any two of the three equations (2.3) - (2.5).

From the pair of equations (1.12) together with (2.3), we have

$$(\rho q)_\phi = \frac{F\Gamma_{11}^2 - E\Gamma_{12}^2}{\sqrt{E}\,W}, \qquad (\rho q)_\psi = \frac{F\Gamma_{12}^2 - E\Gamma_{22}^2}{\sqrt{E}\,W} \tag{2.6}$$

If, using (2.6) and (1.10), the derivatives of q and δ are eliminated from the relation ({1}, eg. 5.2),

$$\sqrt{E}W\omega = Fq_\phi - Eq_\psi + Jq\alpha_\phi ,$$

we get

$$\sqrt{E}W\omega = \frac{EG}{\rho\sqrt{EW}}\Gamma_{11}^2 - \frac{2EF}{\rho\sqrt{EW}}\Gamma_{12}^2 + \frac{E^2}{\rho\sqrt{EW}}\Gamma_{22}^2 - \frac{\sqrt{E}}{\rho^2 W}(F\rho_\phi - E\rho_\psi). \tag{2.7}$$

Using (1.13), this equation may be written as

$$\omega = \frac{1}{W}\left\{\left(\frac{F}{\rho W}\right)_\phi - \left(\frac{E}{\rho W}\right)_\psi\right\}. \tag{2.8}$$

This equation would then replace the vorticity equation of (1.3). We now have a system of equations which may be used to replace (1.3) and equivalently, (1.1).

<u>Theorem 2.1</u>. If the streamlines ψ = const. of the flow of a viscous compressible fluid are taken as one set of coordinates in a curvilinear coordinate system ϕ, ψ in the physical plane, the system (1.3) of five partial differential equations for u, v, ω, p and ε may be replaced by the system

$$G(\rho q q_\phi + p_\phi + \tfrac{4}{3}\mu\varepsilon_\phi) - F(\rho q q_\psi + p_\psi + \omega + \tfrac{4}{3}\mu\varepsilon_\psi) = -\mu J\omega_\psi ,$$

(Navier-Stokes)

$$-F(\rho q q_\phi + p_\phi + \tfrac{4}{3}\mu\varepsilon_\phi) + E(\rho q q_\psi + p_\psi + \omega + \tfrac{4}{3}\mu\varepsilon_\psi) = \mu J\omega_\phi$$

$$\rho q = \frac{\sqrt{E}}{W}, \qquad \left(\frac{1}{\rho}\right)_\phi = \pm\varepsilon W, \qquad \text{(continuity, gradient)}$$

$$\omega = \frac{1}{W}\{ (\frac{F}{\rho W})_{\phi} - (\frac{E}{\rho W})_{\psi} \}, \qquad \text{(vorticity)}$$

and

$$K = \frac{1}{W} \{ (\frac{W}{E} \Gamma_{11}^{2})_{\psi} - (\frac{W}{E} \Gamma_{12}^{2})_{\phi} \} = 0, \qquad \text{(Gauss)}$$

of six equations for E, F, G, q, ω, P, ρ and ε as functions of ϕ and ψ. Once a solution is found for this system, the flow in the physical plane is obtained by the two relations (1.10) and (1.11).

The Gauss equation above insures that the first fundamental form given in (1.7) with E, F and G as functions of ϕ and ψ is for a plane.

We now proceed to a second formulation of the equations given in Theorem 2.1.

If we differentiate the first equation of (2.1) with respect to ψ and the second with respect to ϕ and subtract, we arrive at the equation

$$\mu W \Delta_2 \omega \mp \omega_{\phi} \pm q \frac{\partial(\rho,q)}{\partial(\psi,\phi)} = 0 \tag{2.9}$$

where Δ_2 is Beltrami's differential parameter of second order {4},

$$\Delta_2 f = \frac{1}{W} \{ (\frac{Gf_{\phi} - Ff_{\psi}}{W})_{\phi} + (\frac{-Ff_{\phi} + Ef_{\psi}}{W})_{\psi} \}, \tag{2.10}$$

and $\frac{\partial(\rho,q)}{\partial(\psi,\phi)}$ is the Jacobian $\rho_{\psi}q_{\phi} - \rho_{\phi}q_{\psi}$.

Using the equation (2.3) for q, we may eliminate the derivatives of q from (2.9) to obtain

$$\mu W \Delta_2 \omega \mp \omega_{\phi} \pm \frac{1}{2} \frac{\partial(1/\rho, E/W^2)}{\partial(\phi,\psi)} = 0. \tag{2.11}$$

We will now derive an equation for the pressure p. Using (2.3), we obtain the derivatives of q and with the help of (1.12), (1.13) and (2.7) write them in the form,

$$\rho q q_{\phi} = \pm \varepsilon \frac{E}{W} + \frac{1}{\rho} \{ \frac{F}{W^2} \Gamma_{11}^{2} - \frac{E}{W^2} \Gamma_{12}^{2} \} ,$$

$$\rho q q_{\psi} + \omega = \pm \varepsilon \frac{F}{W} + \frac{1}{\rho} \left\{ \frac{G}{W^2} \Gamma_{11}^{2} - \frac{F}{W^2} \Gamma_{12}^{2} \right\} . \tag{2.12}$$

Then equation (2.2) may be written as,

$$\mu\omega_{\phi} = \pm \frac{-Fp_{\phi}+Ep_{\psi}}{W} \pm \frac{4}{3}\mu \frac{-F\varepsilon_{\phi}+E\varepsilon_{\psi}}{W} \pm \frac{1}{\rho W} \Gamma_{11}^{2} ,$$

$$\mu\omega_{\psi} = \mp \frac{Gp_{\phi}-Fp_{\psi}}{W} \mp \frac{4}{3}\mu \frac{G\varepsilon_{\phi}-F\varepsilon_{\psi}}{W} \pm \frac{1}{\rho W} \Gamma_{12}^{2}$$

We now apply the integrability condition $\omega_{\phi\psi} = \omega_{\psi\phi}$ and use the Gauss equation to obtain

$$\Delta_2 \left(p + \frac{4}{3}\mu\varepsilon\right) = \frac{2}{\rho} \frac{\Gamma_{11}^{2} \Gamma_{12}^{2} - (\Gamma_{12}^{2})^{2}}{W^2} + \left(\frac{1}{\rho}\right)_{\phi} \frac{\Gamma_{12}^{2}}{W^2} - \left(\frac{1}{\rho}\right)_{\psi} \frac{\Gamma_{11}^{2}}{W^2} \tag{2.13}$$

Using (2.8), the vorticity ω may be eliminated from (2.11) and using (2.4), ε may be eliminated from (2.13). Then these two equations, together with the Gauss equation form an underdetermined system of three equations for E, F, G, ρ and p. Once a solution is obtained for this system, the flow in the physical plane is obtained from equations (1.10) and (1.11). We now state the following theorem:

Theorem 2.2. The twodimensional flow of a viscous compressible fluid may be described by first solving the underdetermined system of three partial differential equations given by (2.11) and (2.13) from which ω and ε are eliminated and the Gauss equation $K = 0$ for E, F, G, ρ and p and then obtaining the angle of inclination $\alpha(\phi,\psi)$ from (1.10) and $x(\phi,\psi)$, $y(\phi,\psi)$ from (1.11) which will then implicitly define the streamfunction $\psi(x,y)$.

We will now develop a third system of equations for q, ω, p and ρ as functions of ϕ and ψ. We set

$$\tilde{E} = \mu^2\omega_\phi^{\,2} + \left(\rho q q_\phi + p_\phi - \frac{4}{3}\mu\varepsilon_\phi\right)^2 ,$$

$$\tilde{F} = \mu^2\omega_\phi\omega_\psi + \left(\rho q q_\phi + p_\phi - \frac{4}{3}\mu\varepsilon_\phi\right)\left(\rho q q_\psi + p_\psi + \omega - \frac{4}{3}\mu\varepsilon_\psi\right) , \tag{2.14}$$

$$\tilde{G} = \mu^2\omega^2_\psi + \left(\rho q q_\psi + p_\psi + \omega - \frac{4}{3}\mu\varepsilon_\psi\right)^2 ,$$

$$\tilde{J} = \mu\omega_\phi\left(\rho q q_\psi + p_\psi + \omega - \frac{4}{3}\mu\varepsilon_\psi\right) - \mu\omega_\psi\left(\rho q q_\phi + p_\phi - \frac{4}{3}\mu\varepsilon_\phi\right) .$$

The equations (2.2) may then be written as

$$\frac{E}{\tilde{E}} = \frac{F}{\tilde{F}} = \frac{G}{\tilde{G}} = \frac{J}{\tilde{J}} = \frac{W}{\tilde{W}} \tag{2.15}$$

with $\tilde{W} = \pm\tilde{J}$ according as $W = \pm J$.

From (2.3) and (2.15) we have

$$W = \frac{\tilde{E}}{\rho^2 q^2 \tilde{W}} \tag{2.16}$$

so that equations (2.15) yield

$$E = \frac{\tilde{E}^2}{\rho^2 q^2 \tilde{W}^2} , \quad F = \frac{\tilde{E}\tilde{F}}{\rho^2 q^2 \tilde{W}^2} , \quad G = \frac{\tilde{E}\tilde{G}}{\rho^2 q^2 \tilde{W}^2} , \tag{2.17}$$

which shows that if ω, q, ε, p and ρ are known as functions of ϕ and ψ, the quantities E, F and G may be calculated from (2.17) which would then give the flow in the physical plane from equations (1.10) and (1.11).

The equation (2.5) together with (2.15) will yield us the first equation of the new system.

$$\frac{\tilde{E}}{\tilde{J}} = -\frac{\rho\phi}{\varepsilon}q^2 . \tag{2.18}$$

The vorticity equation (2.8), which can now be written in the form

$$\left(\frac{\tilde{F}}{\rho\tilde{J}}\right)_\phi - \left(\frac{\tilde{E}}{\rho\tilde{J}}\right)_\psi = \frac{\omega}{\rho^2 q^2}\frac{\tilde{E}}{\tilde{J}} . \tag{2.19}$$

The Gauss equation may be obtained from the integrability condition

$$\alpha_{\phi\psi} = \alpha_{\psi\phi} \tag{2.20}$$

where α_ϕ and α_ψ are obtained from (1.10) as

$$\alpha_\phi = \frac{J}{E}\Gamma_{11}^{2} , \qquad \alpha_\psi = \frac{J}{E}\Gamma_{12}^{2} . \tag{2.21}$$

To obtain expressions for the Christoffell symbols Γ_{11}^2 and Γ_{12}^2, we use (2.4) to eliminate ε from (2.12) and use these equations to eliminate the derivatives of q from (2.2). We then have

$$\begin{aligned} \frac{1}{\rho W}\Gamma_{11}^{2} &= \mu\omega_\phi + \left(p_\phi + \tfrac{4}{3}\mu\varepsilon_\phi\right)\frac{F}{W} - \left(p_\psi + \tfrac{4}{3}\mu\varepsilon_\psi\right)\frac{E}{W} , \\ \frac{1}{\rho W}\Gamma_{12}^{2} &= \mu\omega_\psi + \left(p_\phi + \tfrac{4}{3}\mu\varepsilon_\phi\right)\frac{G}{W} - \left(p_\psi + \tfrac{4}{3}\mu\varepsilon_\psi\right)\frac{F}{W} + \left(\frac{1}{\rho}\right)_\phi \frac{1}{W} \end{aligned} \tag{2.22}$$

The integrability condition (2.20) then yields the Gauss equation in the form

$$\begin{aligned} &\left\{\frac{1}{\rho q^2}\left(p_\phi + \tfrac{4}{3}\mu\varepsilon_\phi\right)\frac{\tilde{F}}{\tilde{J}} - \frac{1}{\rho q^2}\left(p_\psi + \tfrac{4}{3}\mu\varepsilon_\psi\right)\frac{\tilde{E}}{\tilde{J}}\right\}_\psi \\ &-\left\{\frac{1}{\rho q^2}\left(p_\phi + \tfrac{4}{3}\mu\varepsilon_\phi\right)\frac{\tilde{G}}{\tilde{J}} - \frac{1}{\rho q^2}\left(p_\psi + \tfrac{4}{3}\mu\varepsilon_\psi\right)\frac{\tilde{F}}{\tilde{J}} + \frac{\varepsilon}{\rho q^2}\right\} = \frac{2\mu}{\rho q^3}\frac{\partial(\omega,q)}{\partial(\phi,\psi)} . \end{aligned} \tag{2.23}$$

<u>Theorem 2.3</u>. When the streamliner ψ = const. of a steady viscous compressible flow are taken as one set of coordinates in a curvilinear coordinate system ϕ, ψ in the physical plane, the system (1.1) for the velocity componants u, v, pressure p and density ρ as functions of x and y may be replaced by a system of three equations (2.18), (2.19) and (2.23) for the vorticity ω, speed q, velocity gradient ε, pressure p and density ρ as functions of ϕ and ψ. Once the solutions for this system are found, the flow in the physical plane may be obtained by calculating the inclination $\alpha(\phi,\psi)$ from (1.10), $x(\phi,\psi)$ and $y(\phi,\psi)$ from (1.11) with the help of (2.17).

The systems of equations developed so far can be made determinate if the equation of state is unknown and by a proper choice of the coordinate curves ϕ = const. For example, if the curves ϕ = const. are chosen as orthogonal to the streamlines, the system given in Theorem 2.3 becomes

$$\tilde{F} = o\ , \quad \frac{\tilde{E}}{\tilde{J}} = \frac{\rho_\phi q^2}{\varepsilon}\ ,$$

$$-\left(\frac{\tilde{E}}{\rho\tilde{J}}\right)_\psi = \frac{\omega}{\rho q}\ \frac{\tilde{E}}{\tilde{J}}\ , \tag{2.24}$$

$$\left\{-\frac{1}{\rho q^2}\left(p_\psi + \frac{4}{3}\mu\varepsilon_\psi\right)\frac{\tilde{E}}{\tilde{J}}\right\}_\psi - \left\{\frac{1}{\rho q^2}\left(p_\phi + \frac{4}{3}\mu\varepsilon_\phi\right)\frac{\tilde{G}}{\tilde{J}} + \frac{\varepsilon}{\rho q^2}\right\}_\phi = \frac{\partial\mu}{\rho q^3}\ \frac{\partial(\omega,q)}{\partial(\phi,\psi)}$$

One may also choose ϕ = p to obtain another system of equations for ω, q, ε and ρ as functions of p and ψ provided the isobars and streamlines do not coincide.

3. Special Cases.

Consider the orthogonal system of curves consisting of tangent lines and the involuter of an arbitrary curve C. The system may be represented by two parameters η = const., ξ = const. where η is the angle of elevation of the tangent lines. If $\sigma = \sigma(\eta)$ is the arc-length of C from a fixed reference point, the radius of curvature is given by

$$\frac{d\sigma}{d\eta} = R \quad , \tag{3.1}$$

and the first fundamental form is given by

$$ds^2 = d\xi^2 + (\xi-\sigma)^2 d\eta^2 . \tag{3.2}$$

We shall first study flows for which the streamlines are tangential to the curve C. We then choose the second coordinate curves ϕ = const. as the involutes of C so that

$$\phi = \phi(\xi), \quad \psi = \psi(\eta). \tag{3.3}$$

Comparing (3.2) with (1.7) we get expressions for ε, F, G and J as

$$\varepsilon = \left(\frac{1}{\dot{\phi}}\right)^2, \; F = 0, \quad G = \left(\frac{\xi-\sigma}{\psi'}\right)^2 \quad , \quad J = \frac{\xi-\alpha}{\dot{\phi}\psi'} \quad , \tag{3.4}$$

where $\dot{} = \dfrac{d}{d\xi}$ and $' = \dfrac{d}{d\eta}$.

From (2.3), (2.4) and (2.8) we have expressions for q, ω and ε as

$$q = \frac{\psi'}{\rho(\xi-\sigma)} \quad , \qquad \varepsilon = \frac{\psi'}{\xi-\sigma}\left(\frac{1}{\rho}\right)_{\xi} \; , \qquad \omega = -\frac{1}{\xi-\sigma}\left\{\frac{\psi'}{\rho(\xi-\sigma)}\right\}_{\eta} . \tag{3.5}$$

The Gauss equation of Theorem 2.2 is identically satisfied and the equation (2.11) for ω becomes

$$a_1(\xi-\sigma)^5 + a_2(\xi-\sigma)^4 + a_3(\xi-\sigma)^3 + a_4(\xi-\sigma)^2 + a_5(\xi-\sigma) + a_6 = 0, \tag{3.6}$$

where

$$a_1 = -\mu\{\psi'(\frac{1}{\rho})_{\xi\xi\eta} + \psi''(\frac{1}{\rho})_{\xi\xi}\} \quad ,$$

$$a_2 = 3\mu\psi'(\frac{1}{\rho})_{\xi\eta} + 3\mu\psi''(\frac{1}{\rho})_{\xi} - \mu\psi'\sigma'(\frac{1}{\rho})_{\xi\xi} + 2\psi'\psi''(\frac{1}{\rho})_{\xi} + \psi'^2(\frac{1}{\rho})_{\xi\eta} \;,$$

$$a_3 = 5\mu\psi'\sigma \;(\frac{1}{\rho})_{\xi} - 4\mu\psi'(\frac{1}{\rho})_{\eta} - 3\mu\psi'''(\frac{1}{\rho})_{\eta} - 3\mu\psi''(\frac{1}{\rho})_{\eta\eta} - \mu\psi'(\frac{1}{\rho})_{\eta\eta\eta}$$

$$-4\mu\psi''(\frac{1}{\rho}) + 2\psi'^2\sigma'(\frac{1}{\rho})_{\xi} - \psi'^2(\frac{1}{\rho})_{\eta} - 2\psi''\psi''(\frac{1}{\rho}) \;,$$

(3.7) $$a_4 = (9\mu\psi'\sigma' - 6\mu\psi''\sigma'' - \mu\psi'\sigma''' - 3\psi'^2\sigma')\frac{1}{\rho}$$ (3.7)

$$-(12\mu\psi''\sigma' + 4\mu\psi'\sigma'')(\frac{1}{\rho})_{\eta} - 6\mu\psi'\sigma'(\frac{1}{\rho})_{\eta\eta} \;,$$

$$a_5 = -15\psi'\sigma'(\frac{1}{\rho})_{\eta} - (15\psi''\sigma'^2 + 10\psi'\sigma'\sigma'')(\frac{1}{\rho}) \;,$$

$$a_6 = -15\psi'\sigma'^3 \;.$$

Equation (3.6) is an identity which must hold everywhere, and it must also be valid on the curve C, given parametrically as $\xi = \sigma(\eta)$. Thus $a_6 \equiv 0$ which could happen only if $\sigma' = 0$. By (3.1) This means that the radius of curvature vanishes and the curve C must reduce to a point. We thus have the theorom,

Theorem 3.1. If the streamlines in the two dimensional flow of a viscous compressible flow are straight lines, they must be either parallel or concurrent.

As a second example we choose the involutes of the curve C as streamlines and the tangent lines as the orthogonal coordinate curves ϕ = const. so that

$$\phi = \phi(\eta) \ , \qquad \psi = \psi(\xi) \ . \tag{3.8}$$

We then have as before

$$E = \frac{(\xi-\sigma)^2}{\phi'^2} \ , \quad F=0 \ , \quad G = \frac{1}{\dot{\psi}^2} \ , \quad J = \frac{\xi-\sigma}{\phi'\dot{\psi}} \tag{3.9}$$

and

$$\rho q = \dot{\psi} \ , \quad \varepsilon = \frac{\dot{\psi}}{\xi-\sigma}\left(\frac{1}{\rho}\right)_\eta \ , \quad \omega = -\left(\frac{\dot{\psi}}{\rho}\right)_\xi - \frac{\dot{\psi}}{\rho(\xi-\sigma)} \ . \tag{3.10}$$

Using (3.9) and (3.10), equation (2.11) may now be written as

$$a_1(\xi-\sigma)^5 + a_2(\xi-\sigma)^4 + a_3(\xi-\sigma)^3 + a_4(\xi-\sigma)^2 + a_5(\xi-\sigma) + a_6 = 0, \tag{3.11}$$

where

$$a_1 = \mu\{\frac{\psi^{(4)}}{\rho} + 3\dddot{\psi}\left(\frac{1}{\rho}\right)_\xi + 3\ddot{\psi}\left(\frac{1}{\rho}\right)_{\xi\xi} + \dot{\psi}\left(\frac{1}{\rho}\right)_{\xi\xi\xi}\} \ ,$$

$$a_2 = 2\mu\frac{\dddot{\psi}}{\rho} + 4\mu\ddot{\psi}\left(\frac{1}{\rho}\right)_\xi + 2\mu\dot{\psi}\left(\frac{1}{\rho}\right)_{\xi\xi} - 2\dot{\psi}\ddot{\psi}\left(\frac{1}{\rho}\right)_\eta - \dot{\psi}^2\left(\frac{1}{\rho}\right)_{\xi\eta}$$

$$a_3 = -\mu\frac{\ddot{\psi}}{\rho} - \mu\dot{\psi}\left(\frac{1}{\rho}\right)_\xi + \mu\ddot{\psi}\left(\frac{1}{\rho}\right)_{\eta\eta} + \mu\dot{\psi}\left(\frac{1}{\rho}\right)_{\xi\eta\eta} - \dot{\psi}^2\left(\frac{1}{\rho}\right)_\eta \tag{3.12}$$

$$a_4 = \mu\dot{\psi}\left(\frac{1}{\rho}\right) + \mu\dot{\psi}\left(\frac{1}{\rho}\right)_{\eta\eta} + \mu\ddot{\psi}\left(\frac{1}{\rho}\right)_\eta\sigma' + \mu\dot{\psi}\left(\frac{1}{\rho}\right)_{\xi\eta}\sigma' - \frac{\dot{\psi}^2\sigma'}{\rho}$$

$$a_5 = \mu\{3\dot{\psi}\left(\frac{1}{\rho}\right)_\eta\sigma' + \frac{\dot{\psi}}{\rho}\sigma''\} \ , \quad \alpha_6 = 3\mu\frac{\dot{\psi}}{\rho}\sigma'^2$$

Again, (3.11) should hold on the curve C, leading to the conclusion that $\sigma' = o$ and the curve must reduce to a point. Thus we have

Theorem 3.2. The streamlines in the two dimensional flow of a compressible viscous fluid can be involutes of a curve C only if C reduces to a point and the streamlines will then be circles concentric at this point.

To obtain the flow in the physical plane, one would need the equation of state $\rho=\rho(p)$. Then equation (2.13) together with either (3.6) or (3.11) would give a determinate system of equations for the pressure p and streamfunction ψ. Once these functions are solved, the flow in the physical plane can be obtained by using (3.4) and (3.9) in (1.10) and (1.11).

REFERENCES

1. MARTIN, M.H., The Flow of a Viscous Fluid I., Arch. Rational Mech. Anal., 41 (1971), pp. 266-288.

2. CHANDNA, O. P. and KALONI, P. N., Steady Plane Flows of Cosserat Fluids, SIAM J. Appl. Math., 31(1976), pp. 686-698.

3. NATH, V. I. and CHANDNA, O. P., On Plane Viscous Magneto hydrodynamic Flows. Quart. Appl. Math., 31(1973), pp. 351-362.

4. EISENHART, L.P., A Treatise on the Differential Geometry of Curves and Surfaces. New York (1909).

LMED

8-8